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The 16th Hilbert problem on algebraic limit cycles

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ABSTRACT

For real planar polynomial differential systems there appeared a simple version of the 16th Hilbert problem on algebraic limit cycles: *Is there an upper bound on the number of algebraic limit cycles of all polynomial vector fields of degree m ?* In [J. Llibre, R. Ramírez, N. Sadovskaia, On the 16th Hilbert problem for algebraic limit cycles, J. Differential Equations 248 (2010) 1401–1409] Llibre, Ramírez and Sadovskaia solved the problem, providing an exact upper bound, in the case of invariant algebraic curves generic for the vector fields, and they posed the following conjecture: *Is $1 + (m - 1)(m - 2)/2$ the maximal number of algebraic limit cycles that a polynomial vector field of degree m can have?*

In this paper we will prove this conjecture for planar polynomial vector fields having only nodal invariant algebraic curves. This result includes the Llibre et al.'s as a special one. For the polynomial vector fields having only non-dicritical invariant algebraic curves we answer the simple version of the 16th Hilbert problem.

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1. Introduction and the statement of the main results

The second part of the 16th Hilbert problem still remains open (see, for example, [8,20]), even through some nice results on the upper bounds of the number of limit cycles can be found in the references (see, for instance, [1,5,9,10] and the references therein). Related to algebraic limit cycles of real planar polynomial vector fields there appeared a simple version of the 16th Hilbert problem, see Llibre et al. [13].

A simple version of the 16th Hilbert problem. *Is there an upper bound on the number of algebraic limit cycles of all real planar polynomial vector fields of a given degree?*

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This simple version of the 16th Hilbert problem provides a nice connection between the two parts of the 16th Hilbert problem.

Consider real planar polynomial vector fields of degree m

$$\mathcal{X} = p(x, y) \frac{\partial}{\partial x} + q(x, y) \frac{\partial}{\partial y}, \quad (1.1)$$

with $p(x, y), q(x, y) \in \mathbb{R}[x, y]$ the ring of real polynomials in x, y and $\max\{\deg p, \deg q\} = m$, or the associated polynomial differential systems

$$\dot{x} = p(x, y), \quad \dot{y} = q(x, y).$$

An algebraic curve $f = 0$ with $f \in \mathbb{C}[x, y]$ the ring of polynomials in x, y with coefficients in \mathbb{C} is *invariant* by the vector field \mathcal{X} if there exists some $K \in \mathbb{C}[x, y]$ such that

$$\mathcal{X}f = p \frac{\partial f}{\partial x} + q \frac{\partial f}{\partial y} = Kf.$$

The polynomial K is called the *cofactor* of f . An *algebraic limit cycle* is a limit cycle which is contained in an invariant algebraic curve of \mathcal{X} . A *limit cycle* of an analytic vector field is an isolated periodic orbit in the set of all periodic orbits of the vector field.

The simple version of the 16th Hilbert problem, i.e. the problem on the upper bound of the number of algebraic limit cycles, is solved in [13] for all real planar polynomial vector fields which have only irreducible invariant algebraic curves generic. A set of invariant algebraic curves, saying $f_j = 0$, $j = 1, \dots, k$, of a planar polynomial vector field is *generic* if the following five conditions hold:

- All the curves $f_j = 0$ are *non-singular* (i.e. there are no points of $f_j = 0$ at which f_j and its first derivative all vanish).
- The highest order homogeneous terms of f_j have no repeated factors.
- If two curves intersect at a point in the affine plane, they are transversal at this point.
- There are no more than two curves $f_j = 0$ meeting at any point in the affine plane.
- There are no two curves having a common factor in the highest order homogeneous terms.

The main result of Llibre et al. [13] proved that *for a real planar polynomial vector field of degree m having all its irreducible invariant algebraic curves generic, the maximal number of algebraic limit cycles is at most $1 + (m - 1)(m - 2)/2$ if m is even, and $(m - 1)(m - 2)/2$ if m is odd, and the upper bounds can be reached.* In the same paper the authors' conjecture 3 stated that

Conjecture. *Is $1 + (m - 1)(m - 2)/2$ the maximal number of algebraic limit cycles that a polynomial vector field of degree m can have?*

Our first result verifies this conjecture for real planar polynomial vector fields having only nodal invariant algebraic curves. We say that an algebraic curve S (not necessary irreducible) is *nodal* if all its singularities are of normal crossing type, that is at any singularity of S there are exactly two branches of S which intersect transversally.

Theorem 1.1. *If a real planar polynomial vector field (1.1) of degree m has only nodal invariant algebraic curves taking into account the line at infinity, then the following hold.*

- The maximal number of algebraic limit cycles of the vector fields is at most $1 + (m - 1)(m - 2)/2$ when m is even, and $(m - 1)(m - 2)/2$ when m is odd.*
- There exist systems of the form (1.1) which have the maximal number of algebraic limit cycles.*

We mention that our result is an essential improvement of that given in [13], because our assumptions only satisfy the third and fourth conditions of the generic conditions of Theorem 1 of [13].

Recently Llibre et al. [14] obtained an upper bound on the number of algebraic limit cycles for real planar polynomial vector fields which have only non-singular invariant algebraic curves. The main result states that *for a real planar polynomial vector field of degree n having all its irreducible invariant algebraic curves non-singular, the maximal number of algebraic limit cycles is at most $n^4/4 + 3n^2/4 + 1$.*

We note that the results given in [13,14] both require a sufficient condition that all the invariant algebraic curves of a prescribed vector field are non-singular, and so they cannot be self-intersected.

Our next result will study the case that the invariant algebraic curves may be singular and the vector field has a more general form than that given in (1.1), i.e.

$$\mathcal{X} = (p(x, y) + xr(x, y))\frac{\partial}{\partial x} + (q(x, y) + yr(x, y))\frac{\partial}{\partial y}, \quad (1.2)$$

where $p, q, r \in \mathbb{R}[x, y]$, $\max\{\deg p, \deg q, \deg r\} = m$ and r is a homogeneous polynomial or is identically zero. We also call m the degree of the vector field (1.2). In the next section we will give more explanation on the degree m . For people working in real planar polynomial vector fields they usually call (1.2) a vector field of degree $m + 1$ if $r(x, y) \not\equiv 0$.

Recall that Theorem 1.1 has the restriction on the singularities of the invariant algebraic curves. We now turn to the case having some assumption on singularities of the vector fields. We assume that the singularities of the vector field on the invariant algebraic curves are non-dicritical. A singularity of a vector field is *non-dicritical* if there are only finitely many invariant integral curves passing through it. An invariant algebraic curve is *non-dicritical* if there is no dicritical singularities on it. Clearly a non-dicritical algebraic curve can be singular.

The following is our second main result.

Theorem 1.2. *If a real planar polynomial vector field (1.2) of degree m has all its invariant algebraic curves non-dicritical, then the following hold.*

- (a) *If $r(x, y) \equiv 0$, the maximal number of algebraic limit cycles of the vector fields is at most $1 + m(m - 1)/2$ when m is even, and $m(m - 1)/2$ when m is odd.*
- (b) *If $r(x, y) \not\equiv 0$, the maximal number of algebraic limit cycles of the vector fields is at most $1 + (m + 1)m/2$ when m is even, and $(m + 1)m/2$ when m is odd.*

We note that Theorem 1.2 solves the simple version of the 16th Hilbert problem on algebraic limit cycles for real planar polynomial vector fields having only non-dicritical invariant algebraic curves. From the proof of this theorem we guess the upper bound is not the best one. We conjecture that *the best upper bound for the number of algebraic limit cycles in the non-dicritical case should be the same as that of Theorem 1.1*. We remark that the invariant algebraic curves in Theorem 1.2 may not satisfy any one of the conditions that the generic algebraic curves have.

Theorem 1.2 has an easy consequence.

Corollary 1.3. *If a real planar polynomial vector field (1.2) of degree m has no dicritical singularities, then the following hold.*

- (a) *If $r(x, y) \equiv 0$, the maximal number of algebraic limit cycles of the vector fields is at most $1 + m(m - 1)/2$ when m is even, and $m(m - 1)/2$ when m is odd.*
- (b) *If $r(x, y) \not\equiv 0$, the maximal number of algebraic limit cycles of the vector fields is at most $1 + (m + 1)m/2$ when m is even, and $(m + 1)m/2$ when m is odd.*

The following result provides an exact upper bound on the number of algebraic limit cycles for polynomial vector fields in the non-dicritical case with an extra assumption.

Theorem 1.4. For real planar polynomial vector fields (1.2) of degree $m \geq 2$ having no dicritical singularities, if they have at least three invariant algebraic curves then the following hold.

- (a) The maximal number of algebraic limit cycles of the vector fields is at most $1 + (m - 1)(m - 2)/2$ when m is even, and $(m - 1)(m - 2)/2$ when m is odd.
- (b) The maximal number can be reached only for some polynomial vector fields (1.2) with $r(x, y) \not\equiv 0$ and the number of invariant algebraic curves to be three.

Theorem 1.4 has verified Conjecture 3 of [13] in the non-dicritical case with the extra assumption on the number of invariant algebraic curves. Its proof follows from those of Theorems 1.1 and 1.2, the details are omitted.

This paper is organized as follows. In the next section we will present some backgrounds on the degree of invariant algebraic curves for holomorphic singular foliations. In Section 3 we will prove our main results. The last section is an appendix, which provides a proof to Proposition 2.3.

2. Upper bound on the degree of invariant algebraic curves

Let \mathcal{F} be a holomorphic singular foliation by curves of the complex projective plane $\mathbb{CP}(2)$. Take an affine coordinate system (x, y) such that \mathcal{F} are the solutions of $\tilde{P} dy - \tilde{Q} dx = 0$. Let L be a straight line which is not invariant by \mathcal{F} . Then the maximal number of the points $p \in L$ such that either $p \in \{(x, y); P(x, y) = Q(x, y) = 0\}$ or the leaf of \mathcal{F} through p is tangent to L is bounded by $\max\{\deg P, \deg Q\}$. For a generical line L , this maximal number is a constant. We call it the degree of \mathcal{F} .

Consider a holomorphic singular foliation \mathcal{F} of degree m . In the projective coordinates, \mathcal{F} can be written as the closed one-form

$$\tilde{\omega} = P(X, Y, Z) dX + Q(X, Y, Z) dY + R(X, Y, Z) dZ,$$

where $P, Q, R \in \mathbb{C}[X, Y, Z]$ are homogeneous polynomials of degree $m + 1$ satisfying the projective condition $XP + YQ + ZR = 0$. As usual, $\mathbb{C}[X, Y, Z]$ denotes the complex polynomial ring in the homogeneous coordinates X, Y and Z . In the affine coordinates, \mathcal{F} can be written as the one-form

$$\omega = -(q(x, y) + yr(x, y)) dx + (p(x, y) + xr(x, y)) dy,$$

or as the vector field

$$\mathcal{X} = (p(x, y) + xr(x, y)) \frac{\partial}{\partial x} + (q(x, y) + yr(x, y)) \frac{\partial}{\partial y},$$

where $p, q, r \in \mathbb{C}[x, y]$ with $\max\{\deg p, \deg q, \deg r\} = m$ and $r(x, y)$ is a homogeneous polynomial of degree m or is naught. If $r \equiv 0$ then $\max\{\deg p, \deg q\} = m$. These claims can be found in [11] and [3].

A point $(X_0, Y_0, Z_0) \in \mathbb{CP}(2)$ is called a *singularity* of \mathcal{F} if $P(X_0, Y_0, Z_0) = Q(X_0, Y_0, Z_0) = R(X_0, Y_0, Z_0) = 0$; or in the affine plane $(X_0, Y_0, Z_0) = (x_0, y_0, 1)$ satisfies $p(x_0, y_0) + x_0 r(x_0, y_0) = q(x_0, y_0) + y_0 r(x_0, y_0) = 0$. A singularity of \mathcal{F} is called *non-dicritical* if there are only finitely many integral curves passing through it. Otherwise, it is called *dicritical*.

An algebraic curve S defined by a reduced homogeneous polynomial $F(X, Y, Z) \in \mathbb{C}[X, Y, Z]$ is called *invariant* by \mathcal{F} if $\tilde{\omega} \wedge dF = F\theta$, where θ is a two-form. Recall that a *reduced polynomial* is the one which has no repeat factors. In what follows, for simplicity we also say F is an invariant algebraic curve. It is easy to prove [23] that F is an invariant algebraic curve if and only if $\mathcal{X}f = kf$ for some $k \in \mathbb{C}[x, y]$, where $f = F|_{Z=1}$.

Theorem 1 of Cerveau and Lins Neto [3] in 1991 obtained the exact upper bound on the degree of nodal invariant algebraic curves, which is the key point to prove Theorem 1.1.

Theorem 2.1. (See Cerveau and Lins Neto [3].) Let \mathcal{F} be a foliation in $\mathbb{CP}(2)$ of degree m , having S as a nodal invariant algebraic curve with the reduced homogeneous equation $F = 0$ of degree n . Then $n \leq m + 2$. Moreover if $n = m + 2$ then F is reducible and the foliation \mathcal{F} is of logarithmic type, that is given by a rational closed form $\sum_i \lambda_i \frac{dF_i}{F_i}$, where $\lambda_i \in \mathbb{C}$ and F_i are the irreducible homogeneous components of F and $\sum_i \lambda_i \deg F_i = 0$.

In the non-dicritical case Carnicer [2] in 1994 obtained the same upper bound as that given in Theorem 2.1, which solved the Poincaré problem [18] in the non-dicritical case. We will use it to prove our Theorem 1.2.

Theorem 2.2. (See Carnicer [2].) Let \mathcal{F} be a holomorphic singular foliation of degree m in $\mathbb{CP}(2)$. Assume that S is an algebraic curve which is invariant by \mathcal{F} , and is given by a reduced polynomial F of degree n . If there are no dicritical singularities of \mathcal{F} on S , then $n \leq m + 2$.

In the proof of this last result, the author had used the following result, which is due to Cerveau and Lins Neto [3].

Proposition 2.3. Assume that \mathcal{F} is a holomorphic singular foliation of degree m in $\mathbb{CP}(2)$, and that S is a reduced algebraic curve of degree n which is invariant by \mathcal{F} . Let $\chi(S)$ be the intrinsic Euler characteristic of S (see [11]) and let $g(S)$ be the topological genus of S . Then

$$\chi(S) = 2 - 2g(S) = \sum_B \mu_p(\mathcal{F}, B) - n(m - 1), \quad (2.1)$$

where the sum runs over all the local branches B of S passing through the singularities of \mathcal{F} in S , and $\mu_p(\mathcal{F}, B)$ is the multiplicity of \mathcal{F} at B passing through the singularity p .

Since the proof of the last result given in [3] has a gap inside, we will present a new proof to it in Appendix A. The multiplicity of \mathcal{F} at B passing through p is defined as follows: for each singularity p of \mathcal{F} such that $p \in S$, and each local branch B of S passing through p , take a vector field $\mathcal{X} = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y}$ to represent \mathcal{F} in a neighborhood of p and a minimal Puiseux's parameterization of B , saying that $\phi = (\phi_1, \phi_2) : \mathbb{D} \rightarrow \mathbb{C}^2$ such that $\phi(0) = 0$, where \mathbb{D} is a disk centered at $0 \in \mathbb{C}$. We define the multiplicity of \mathcal{F} at B to be the order of $\phi^*(\mathcal{X}) = R(t) \frac{d}{dt}$ at $t = 0 \in \mathbb{D}$, denoted by $\mu_p(\mathcal{F}, B)$. Then

$$\mu_p(\mathcal{F}, B) = \frac{1}{2\pi i} \int_{\gamma(B)} \frac{dR(t)}{R(t)},$$

where $\gamma(B) = re^{i\theta}$, $r > 0$ small, is the homology class in $H_1(B \setminus \{p\}, t)$ of the curve $\theta \rightarrow \phi(re^{i\theta})$, $0 \leq \theta \leq 2\pi$.

As a by-product of the equality (2.1) we have the following well-known result. Since the proof is short, we will present it in Appendix A.

Corollary 2.4. An irreducible non-singular algebraic curve S of degree n has the Euler characteristic $\chi(S) = -n(n - 3)$.

3. Proof of the main results

For proving the theorems we need the following Harnack's theorem, for a proof see, for instance, [6,21,22].

Theorem 3.1 (Harnack Theorem). *The number of ovals of a real irreducible algebraic curve of degree n is at most*

$$1 + (n-1)(n-2)/2 - \sum_p v_p(S)(v_p(S)-1),$$

if n is even, or

$$(n-1)(n-2)/2 - \sum_p v_p(S)(v_p(S)-1),$$

if n is odd, where p runs over all the singularities of \mathcal{F} on S , and $v_p(S)$ is the order of S at the singular point p . Moreover these upper bounds can be reached for convenient algebraic curves of degree n .

The following result, due to Giacomini, Llibre and Viano [7], provides the location of limit cycles for a real planar differential system having an inverse integrating factor, for a different proof see [15].

Theorem 3.2. *Let \mathcal{X} be a C^1 vector field defined in the open subset U of \mathbb{R}^2 , and let $V : U \rightarrow \mathbb{R}$ be an inverse integrating factor of \mathcal{X} . If γ is a limit cycle of \mathcal{X} , then γ is contained in $\{(x, y) \in U : V(x, y) = 0\}$.*

3.1. Proof of Theorem 1.1

(a) Write system (1.1) in the one-form

$$q(x, y) dx - p(x, y) dy.$$

Its projective one-form is

$$\omega_0 = ZQ dX - ZP dY + (YP - XQ) dZ, \quad (3.1)$$

where X, Y, Z are the homogeneous coordinates and

$$P = Z^m p(X/Z, Y/Z), \quad Q = Z^m q(X/Z, Y/Z).$$

Consider the holomorphic singular foliation \mathcal{F}_0 induced by the one-form ω_0 . Clearly \mathcal{F}_0 has the infinity as an invariant line. Under the assumption of the theorem, we get from Theorem 2.1 that the total degree n of all invariant algebraic curves of the foliation \mathcal{F}_0 is no more than $m+2$.

Case 1. $n = m+2$. Theorem 2.1 shows that F is reducible, saying $F = F_1 \cdots F_k$ the irreducible decomposition with $k \geq 2$. The one-form ω_0 has the expression

$$\omega_0 = F \sum_{i=1}^k \lambda_i \frac{dF_i}{F_i},$$

where $\lambda_i \in \mathbb{C}$. Recall that an invariant algebraic curve of a real system can be complex. If it happens its conjugate is also an invariant algebraic curve of the system. The one-form ω_0 has the inverse integrating factor F , and consequently is Darboux integrable with the Darboux first integral $H(X, Y, Z) = F_1^{\lambda_1} \cdots F_k^{\lambda_k}$. For more information on the Darboux theory of integrability, see, for instance, [12,16,17].

Since the one-form ω_0 is projective, i.e. $i_E \omega_0 = 0$, where $E = X \frac{\partial}{\partial X} + Y \frac{\partial}{\partial Y} + Z \frac{\partial}{\partial Z}$ is the radial vector field and i_E is the interior product by E , we should have $\lambda_1 \deg F_1 + \cdots + \lambda_k \deg F_k = 0$.

If $k = 2$, the foliation \mathcal{F}_0 has a rational first integral

$$H(X, Y, Z) = F_1^k F_2^{-l}, \quad \text{with } k, l \in \mathbb{N}, (k, l) = 1, \text{ and } k/l = \deg F_2 / \deg F_1.$$

In this case there are infinitely many invariant algebraic curves. Of course they are not possible of nodal type. Otherwise it is in contradiction with Theorem 2.1. So we must have $k \geq 3$.

For $k \geq 3$, we get from the Harnack theorem that each invariant algebraic curve has at most $(\deg F_i - 1)(\deg F_i - 2)/2 + a_i$ ovals, where $a_i = 1$ if $\deg F_i$ is even, and $a_i = 0$ if $\deg F_i$ is odd. So the total number of ovals contained in F_i for $i = 1, \dots, k$ is no more than

$$\sum_{i=1}^k \left(\frac{(\deg F_i - 1)(\deg F_i - 2)}{2} + a_i \right) \leq \frac{(m+2-k)(m+1-k)}{2} + \sum_{i=1}^k a_j,$$

where we have used Lemma 6 of [13] and $\deg F_1 + \dots + \deg F_k = m+2$. Furthermore the equality holds if and only if one of the F_i 's has the degree $m+3-k$ and the others all have degree 1.

Set

$$M(k) = \frac{(m+2-k)(m+1-k)}{2} + \sum_{j=1}^k a_j.$$

Then the maximum of the $M(k)$ for $k \in \{3, \dots, m+2\}$ takes place when $k = 3$, because $\sum_{j=1}^k a_j \leq [m/2] + 1$, where $[\cdot]$ denotes the integer part function. For $k = 3$ and the three invariant algebraic curves have respectively the degrees 1, 1 and m , the maximum is

$$\frac{(m-1)(m-2)}{2} + a,$$

where $a = 1$ if m is even and $a = 0$ if m is odd.

Case 2. $n \leq m+1$. Recall that the line at infinity is invariant by the foliation \mathcal{F}_0 . If $n \leq m$ then the total degree of the invariant algebraic curves in the affine plane is less than m . By the Harnack theorem we get from the proof of Case 1 that the number of algebraic limit cycles is less than the maximal value.

If $n = m+1$, the total degree of the invariant algebraic curves in the affine plane is m . By the Harnack theorem the number of algebraic limit cycles is less than or equal to the maximal value. This proves statement (a).

(b) We only need to prove that there exists a real planar polynomial system of the form (1.1) with degree m which has the maximal number of algebraic limit cycles and the total degree of the invariant algebraic curves in the affine plane is m and $m+1$ respectively, because the line at infinity is invariant.

Case 1. The number $m+1$ is the total degree of the invariant algebraic curves in the affine plane. By the Harnack theorem there exists a non-singular algebraic curve of degree m which has the maximal number, i.e. $(m-1)(m-2)/2 + a$, of ovals, where $a = 1$ if m is even, or $a = 0$ if m is odd. Denote by f_1 this curve. Choose a straight line, called f_2 , as the line at infinity in such a way that it is outside the ovals of f_1 and intersects f_1 transversally. Choose another straight line, called f_3 , which is outside the ovals of f_1 and intersects f_1 and f_2 transversally and does not meet the intersection points of f_1 and f_2 .

Let F_1, F_2 and F_3 be the projectivization of f_1, f_2 and f_3 , respectively. Take $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ non-zero such that $\lambda_1 m + \lambda_2 + \lambda_3 = 0$ and $\lambda_i / \lambda_j \notin \{r \in \mathbb{Q}; r < 0\}$. Then the foliation \mathcal{F}_m induced by the projective one-form $\lambda_1 F_2 F_3 dF_1 + \lambda_2 F_1 F_3 dF_2 + \lambda_3 F_1 F_2 dF_3$ has only the three invariant algebraic curves F_1, F_2, F_3 . Hence \mathcal{F}_m has exactly $(m-1)(m-2)/2 + a$ algebraic limit cycles. In fact \mathcal{F}_m has the

inverse integrating factor $F_1 F_2 F_3$. By Theorem 3.2 \mathcal{F}_m has no other limit cycles, i.e. the non-algebraic ones.

We note that \mathcal{F}_m is a holomorphic singular foliation of degree m . Since it has the line at infinity invariant, its affine expression should be a polynomial differential system of degree m having the form (1.1).

Case 2. The number m is the total degree of the invariant algebraic curves in the affine plane. In fact the proof can be obtained from [4,13]. For completeness and because it is short, we present it here for readers' convenience.

By the Harnack theorem there exists a non-singular algebraic curve of degree m which has the maximal number, i.e. $(m-1)(m-2)/2 + a$, of ovals, where either $a = 1$ or $a = 0$ if m is either even or odd. Denote by $g(x, y)$ this non-singular algebraic curve. Choose a linear function $h(x, y)$ such that $h = 0$ does not intersect the ovals of $g = 0$, and choose $a, b \in \mathbb{R}$ satisfying $ah_x + bh_y \neq 0$, then the real planar differential system

$$\dot{x} = ag - hg_y, \quad \dot{y} = bg + hg_x \quad (3.2)$$

is of degree m and has all the ovals of $g = 0$ as hyperbolic limit cycles. Moreover system (3.2) has no other limit cycles. This proves statement (b) and consequently the theorem.

3.2. Proof of Theorem 1.2

Write system (1.2) in the one-form

$$(q(x, y) + yr(x, y))dx - (p(x, y) + xr(x, y))dy.$$

Its projective one-form is

$$\omega_1 = (ZQ + YR)dX - (ZP + XR)dY + (YP - XQ)dZ, \quad (3.3)$$

where X, Y, Z are the homogeneous coordinates and

$$P = Z^m p(X/Z, Y/Z), \quad Q = Z^m q(X/Z, Y/Z), \quad R = Z^m r(X/Z, Y/Z).$$

Let \mathcal{F}_1 be the holomorphic singular foliation induced by ω_1 . By the assumption of the theorem \mathcal{F}_1 has all the invariant algebraic curves non-dicritical, and their total degree is less than or equal to $m + 2$ by Theorem 2.2.

(a) If $r(x, y) \equiv 0$, the line at infinity is invariant by the foliation \mathcal{F}_1 . So it follows from the proof of Theorem 1.1 that the total degree n of all the invariant algebraic curves in the affine plane is at most $m + 1$. Recall that m is the degree of the polynomial vector field. From the proof of Case 1 of statement (b) of Theorem 1.1, we know that there is a foliation of degree m which has invariant algebraic curves with the total degree $m + 2$ taking into account the line at infinity. Of course, it is reducible that the invariant algebraic curves by the foliation constructed in Case 1 of the proof of statement (b) of Theorem 1.1.

If \mathcal{F}_1 has an irreducible invariant algebraic curve of degree $m + 1$ in the affine plane with the maximal number of ovals that an algebraic curve of degree $m + 1$ can have by Theorem 3.1, then the foliation has the maximal number of algebraic limit cycles. In all the other cases there is not a system of the form (1.2) which has the maximal number of algebraic limit cycles. This proves statement (a).

(b) If $r(x, y) \not\equiv 0$, the line at infinity is not invariant by \mathcal{F}_1 . We get from Theorem 2.2 that the total degree n of all invariant algebraic curves of (1.2) in the affine plane is at most $m + 2$. We claim that there exists a system of the form (1.2) having degree m with $r(x, y) \not\equiv 0$ which has invariant algebraic curves of total degree $m + 2$.

We now prove the claim. Let $f_1, \dots, f_k \in \mathbb{C}[x, y]$ with $k \geq 3$ be reduced such that $\deg f_1 + \dots + \deg f_k = m + 2$ and their projective curves in $\mathbb{CP}(2)$ defined by F_1, \dots, F_k the projectivization of f_1, \dots, f_k are non-singular and intersect transversally and no more than two curves meeting at the same point. Take $\lambda_1, \dots, \lambda_k \in \mathbb{C}$ non-zero such that $\lambda_1 \deg F_1 + \dots + \lambda_k \deg F_k = 0$ and $\lambda_i/\lambda_j \notin \{r \in \mathbb{Q}; r < 0\}$ for $1 \leq i \neq j \leq k$. Then the foliation \mathcal{F}^* induced by the projective one-form $\omega^* = \sum_{j=1}^k \lambda_j (\prod_{i=1, i \neq j}^k F_i dF_j)$ has degree m and has only the invariant algebraic curves defined by F_1, \dots, F_k . Furthermore all the singularities of \mathcal{F}^* are non-dicritical [19], because they are the intersection points of F_1, \dots, F_k and the invariant curves passing through these singularities are only the branches of F_i for $i = 1, \dots, k$. By Theorem 2.2 the total degree of all invariant algebraic curves by \mathcal{F}^* is at most $m + 2$. While f_1, \dots, f_k have the total degree $m + 2$. This implies that the line at infinity of \mathcal{F}^* is not invariant. So its affine expression of \mathcal{F}^* must have the form (1.2) with $r(x, y) \neq 0$. This proves the claim.

If \mathcal{F}_1 has an irreducible invariant algebraic curve of degree $m + 2$ with the maximal number of ovals that an algebraic curve of degree $m + 2$ can have by Theorem 3.1, then the foliation has the maximal number of algebraic limit cycles. In all the other cases there is not a system of the form (1.2) which has the maximal number of algebraic limit cycles.

We complete the proof of the theorem.

We mention that the foliation \mathcal{F}^* of degree m constructed in the proof of statement (b) of Theorem 1.2 has at least three invariant algebraic curves with the total degree $m + 2$. We do not know if there is a holomorphic singular foliation of degree m which has a non-dicritical irreducible invariant algebraic curve of degree either $m + 1$ or $m + 2$. Of course as shown in Theorem 2.1 it is not possible for nodal invariant algebraic curves. We guess it is also not possible for non-dicritical invariant algebraic curves, but we cannot prove it now.

Finally we provide an easy example showing the foliation \mathcal{F}^* mentioned above.

Example. For an algebraic curve S in $\mathbb{CP}(2)$ which has the affine representation $f = xy(y - x - 1)$, the projective homogeneous form of f is $F = XY(Y - X - Z)$. Then the holomorphic foliation \mathcal{F}_3^* given by the one-form

$$Y(\lambda_1 Y + \lambda_2 X - \lambda_1 Z) dX - X(\lambda_1 Y + \lambda_2 X + \lambda_2 Z) dY - \lambda_3 XY dZ$$

has degree 1 and has only the invariant algebraic curves F provided that $\lambda_1 + \lambda_2 + \lambda_3 = 0$, $\lambda_i \neq 0$ for $i = 1, 2, 3$ and λ_i/λ_j for $1 \leq i \neq j \leq 3$ non-negative rational numbers. The line at infinity, i.e. $Z = 0$, is not invariant for \mathcal{F}_3^* . The singularities of \mathcal{F}_3^* are non-dicritical, see [19].

Appendix A

A.1. Proof of Proposition 2.3

Since S is an invariant algebraic curve of degree n , we can choose an affine coordinate system (x, y) such that S cuts the line at infinity l_∞ transversely at exactly n points. Let $\mathcal{X} = P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y}$ represent \mathcal{F} in this coordinate system. Without loss of generality, we suppose that $p = (1 : 0 : 0)$ belongs to $S \cap l_\infty$. Making the change of variables $u = \frac{y}{x}$, $v = \frac{1}{x}$, the vector field \mathcal{X} becomes

$$\tilde{\mathcal{X}} = v^{-m+1} \left[(-u\tilde{P}(u, v) + \tilde{Q}(u, v)) \frac{\partial}{\partial u} - v\tilde{P}(u, v) \frac{\partial}{\partial v} \right]$$

where $\tilde{P}(u, v) = v^m P(\frac{1}{v}, \frac{u}{v})$, $\tilde{Q}(u, v) = v^m Q(\frac{1}{v}, \frac{u}{v})$.

In the coordinate system (u, v) , since S intersects l_∞ transversely we can take $u = \psi(v)$ as the local branch B_∞ of S passing through the singularity $(1 : 0 : 0)$. Clearly, ψ is analytic in v . Using the change of variables $\alpha = u - \psi(v)$, $\beta = v$, the vector field $\tilde{\mathcal{X}}$ can be written as

$$\tilde{\mathcal{X}}^* = \beta^{-m+1} \left[-(\alpha + \psi(\beta)) \tilde{P}^* + \tilde{Q}^* + \beta \psi'(\beta) \tilde{P}^* \right] \frac{\partial}{\partial \alpha} - \beta \tilde{P}^* \frac{\partial}{\partial \beta},$$

where $\tilde{P}^* = \tilde{P}(\alpha + \psi(\beta), \beta)$, $\tilde{Q}^* = \tilde{Q}(\alpha + \psi(\beta), \beta)$. Since $h(u, v) = u - \psi(v)$ is an analytic solution of $\tilde{\mathcal{X}}$, there exists a locally analytic function $k(u, v)$ such that

$$(-u \tilde{P}(u, v) + \tilde{Q}(u, v)) \frac{\partial h}{\partial u} - v \tilde{P}(u, v) \frac{\partial h}{\partial v} = hk.$$

Hence, we have

$$-(\alpha + \psi(\beta)) \tilde{P}^* + \tilde{Q}^* + \beta \tilde{P}^* \psi'(\beta) = \tilde{k}^* \alpha,$$

where $\tilde{k}^* = k(\alpha + \psi(\beta), \beta)$. This shows that on B_∞

$$\tilde{\mathcal{X}}^* = \beta^{-m+1} \left(\tilde{k}^* \alpha \frac{\partial}{\partial \alpha} - \beta \tilde{P}^* \frac{\partial}{\partial \beta} \right).$$

Set $\tilde{P}^*|_{\alpha=0} = \beta^l \hat{P}^*(\beta)$ such that $\hat{P}^*(0) \neq 0$, and set

$$\zeta = \frac{\hat{P}^*}{\beta^{m-2-l}} = \frac{\hat{P}^*}{|\beta|^{2(m-2-l)}} \bar{\beta}^{m-2-l},$$

where $\bar{\beta}$ denotes the conjugacy of β . Then the vector field $\tilde{\mathcal{X}}^*|_{\alpha=0}$ at $\beta = 0$ has the multiplicity or a pole of order

$$\frac{1}{2\pi i} \int_{\gamma} \frac{d\zeta}{\zeta} = \frac{m-2-l}{2\pi i} \int_{\gamma} \frac{d\bar{\beta}}{\bar{\beta}} = -(m-2-l),$$

where γ is the homology class in $H_1(B_\infty, \beta)$ of the curve $\theta \rightarrow \phi(re^{i\theta})$ on $\alpha = 0$, $0 \leq \theta \leq 2\pi$. Moreover, from the expression of $\tilde{\mathcal{X}}^*$ we can get easily that $\mu_p(\mathcal{X}, B_\infty) = l + 1$.

Let $\pi: \tilde{S} \rightarrow S$ be a resolution of S by blowing-ups at the singularities of S . Then \tilde{S} is smooth and $2 - 2g(S) = \chi(\tilde{S})$, which is the Euler characteristic of \tilde{S} . We define the intrinsic Euler characteristic $\chi(S)$ to be $\chi(\tilde{S})$, and the vector field in \tilde{S} associated with \mathcal{X} to be $\pi^*(\mathcal{X}|_S) = \tilde{\mathcal{X}}$. For each singularity p of \mathcal{F} in S , and each local branch B of S passing through p , we obtain a singularity \tilde{p} of $\tilde{\mathcal{X}}$ in \tilde{S} and a unique local branch \tilde{B} passing through \tilde{p} which is invariant by $\tilde{\mathcal{X}}$. Then the Poincaré–Hopf index of $\tilde{\mathcal{X}}$ with respect to \tilde{B} at \tilde{p} is $\mu_p(\mathcal{F}, B)$.

From the choice of the local coordinate system at the beginning of the proof of this proposition, we know that $l_\infty \cap S$ contains n points, denoted by p_i , $i = 1, \dots, n$. We denote by l_i associated to p_i the quantity l in the above proof for the singularity p . Then we get from the Poincaré–Hopf Index Theorem that

$$\chi(S) = \sum_B \mu_p(\mathcal{F}, B) - \sum_{i=1}^n (m-2-l_i),$$

where B is taken over all the local branches of S passing through the singularities at the finite plane. Since $\mu_{p_i}(\mathcal{X}, B_\infty) = l_i + 1$ we have

$$\chi(S) = \sum_B \mu_p(\mathcal{F}, B) - n(m-1),$$

where B is taken over all the local branches of S passing through the singularities. We complete the proof of the proposition.

Next we provide some examples showing the application of Proposition 2.3.

Example A.1. Consider the foliation \mathcal{F}_1 of $\mathbb{CP}(2)$ given by the homogeneous differential form

$$\alpha YZ dX + \beta XZ dY - (\alpha + \beta)XY dZ,$$

with $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and $\frac{\alpha}{\beta} \notin \mathbb{R}$ (this assures that all the singularities of \mathcal{F}_1 are non-dicritical). The line $X = 0$ is invariant by the foliation \mathcal{F}_1 , on which there are two singularities: $P_1 = (0 : 1 : 0)$ and $P_2 = (0 : 0 : 1)$. The vector field associated with $\mathcal{F}_1|_{X=0}$ at P_1 is $\alpha z \frac{\partial}{\partial z}$, so $\mu_{P_1}(\mathcal{F}_1, X = 0) = 1$. Similarly, the vector field associated with $\mathcal{F}_1|_{X=0}$ at P_2 is $\alpha y \frac{\partial}{\partial y}$, so $\mu_{P_2}(\mathcal{F}_1, X = 0) = 1$. In addition we have $\chi(X = 0) = 2$. Since the foliation is of degree 1, this verifies the proposition.

Example A.2. Consider the foliation \mathcal{F}_2 of $\mathbb{CP}(2)$ given by the homogeneous differential form

$$(2YZ - X^2)Z dX + X(Y + Z)Z dY + (X^3 - XY^2 - 3XYZ) dZ.$$

The foliation \mathcal{F}_2 has $X = 0$ as an invariant line, which contains exactly two singularities of \mathcal{F}_2 : $P_1 = (0 : 1 : 0)$ and $P_2 = (0 : 0 : 1)$. We can check easily that P_1 and P_2 are both non-dicritical. The vector fields associated with $\mathcal{F}_2|_{X=0}$ at P_1 and P_2 are $-2z^2 \frac{\partial}{\partial z}$ and $-2y \frac{\partial}{\partial y}$, respectively. So we have $\mu_{P_1}(\mathcal{F}_2, X = 0) = 2$ and $\mu_{P_2}(\mathcal{F}_2, X = 0) = 1$. Now the foliation has degree 2, this verifies the proposition.

We note that P_1 and P_2 are respectively dicritical and non-dicritical singularities of \mathcal{F}_2 .

Example A.3. Consider the foliation \mathcal{F}_3 of $\mathbb{CP}(2)$ given by the homogeneous differential form

$$(X^3 - 2Y^2Z)Z dX - X(Y^2 + Z^2)Z dY - (X^4 - 2XY^2Z - XYZ^2 - XY^3) dZ.$$

The foliation \mathcal{F}_3 has also $X = 0$ as an invariant line, on which there are only the non-dicritical singularities: $P_1 = (0 : 1 : 0)$ and $P_2 = (0 : 0 : 1)$. The vector fields associated with $\mathcal{F}_3|_{X=0}$ at P_1 and P_2 are $-2z^2 \frac{\partial}{\partial z}$ and $-2y^2 \frac{\partial}{\partial y}$, respectively. So we have $\mu_{P_1}(\mathcal{F}_3, X = 0) = 2$ and $\mu_{P_2}(\mathcal{F}_3, X = 0) = 2$. Now the foliation has degree 3, this verifies the proposition.

We can check that P_1 and P_2 are both dicritical singularities of \mathcal{F}_3 , in fact they are saddle node.

A.2. Proof of Corollary 2.4

Take an affine coordinate system (x, y) of $\mathbb{CP}(2)$ such that S intersects the line at infinity transversally. Denote by p_i and B_i , $i = 1, \dots, n$, the n intersection points and the n branches of S passing through p_i , respectively. Let $f \in \mathbb{C}[x, y]$ be a reduced equation of the affine part of S . We denote by \mathcal{G}_f the holomorphic foliation by curves of $\mathbb{CP}(2)$ which extends the foliation of \mathbb{C}^2 given by df . Then \mathcal{G}_f has degree $n - 1$. Applying the formula (2.1) to the foliation \mathcal{G}_f , we have

$$\sum_{i=1}^n \mu_{p_i}(\mathcal{G}_f, B_i) = \chi(S) + n(n - 2).$$

Since S is non-singular, it follows that p_i for $i = 1, \dots, n$ are the only singularities of \mathcal{G}_f , which are located at the intersection of S with l_∞ . Moreover we have $\mu_{p_i}(\mathcal{G}_f, B_i) = 1$ for $i = 1, \dots, n$. This shows that $n = \chi(S) + n(n - 2)$, and consequently the corollary follows.

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